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**LETTER TO THE EDITOR**

# Bi-differential calculi and bi-Hamiltonian systems

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**Abstract.** We discuss the relationship between the analysis of completely integrable systems using bi-differential calculi which was introduced by Dimakis and Müller-Hoissen (2000 Bi-differential calculi and integrable models *J. Phys. A: Math. Gen.* **33** 957–74), and the bi-Hamiltonian formalism, in the finite-dimensional case.

In a recent paper in this journal [1] Dimakis and Müller-Hoissen have shown how to generate conservation laws in completely integrable systems by using a bi-differential calculus. In the concluding section of their paper they ask how their approach 'is related to various other characterizations of completely integrable systems', and mention the bi-Hamiltonian formalism as one of these other approaches. We will briefly discuss aspects of the relationship between their work and the bi-Hamiltonian formalism in the finite-dimensional case.

We will be concerned with bi-differential calculi over the exterior algebra $\Omega(\mathcal{A}) = \bigwedge(M)$ on a manifold $M$, where $\mathcal{A} = C^\infty(M)$ is the algebra of real-valued $C^\infty$ functions on $M$, and where one of the derivations is the exterior derivative $d$. (Actually, Dimakis and Müller-Hoissen denote the derivation which plays the role of the exterior derivative here by $\delta$; we have thought it better to stick to the standard notation of differential geometry.) The second derivation $\delta$, which creates the bi-differential calculus, is required to be, like $d$, a derivation of degree 1 of the exterior algebra and to satisfy

$$\delta^2 = 0 \qquad d\delta + \delta d = 0.$$

Our first observation is that, according to Frölicher–Nijenhuis theory, a derivation of degree 1 which (anti-)commutes with $d$ (that is, a derivation of type $d_*$ in the terminology of Frölicher and Nijenhuis) must be of the form $\delta = d_R$ for some type (1, 1) tensor field $R$ on $M$, and that the necessary and sufficient condition for $d_R$ to satisfy ${d_R}^2 = 0$ is that the torsion, or Nijenhuis tensor, of $R$ must be zero. Thus in this particular case, bi-differential calculi are in one-to-one correspondence with type (1, 1) tensor fields with vanishing torsion.

In what follows we will be concerned mainly with the action of $d_R$ on $C^\infty(M)$, for which we have the formula $d_R f = R^*(df)$, where we think of the tensor $R$ as a homomorphism of the module of vector fields on $M$, and $R^*$ as its adjoint acting on 1-forms. In fact a derivation $\delta$ of type $d_*$ is determined by its action on functions—the condition $d\delta + \delta d = 0$ defines its action on $s$-forms for $s \geqslant 1$—and it is easy to see that if $\delta$ is of degree 1 its action on functions must be given by $\delta f = R^*(df)$ for some $R$.

The basic step in the construction of Dimakis and Müller-Hoissen is to define inductively a sequence of $(s-1)$-forms $\chi^{(m)}$, $m = 0, 1, 2, \ldots$, where $s$ is an integer for which closed $s$-forms are exact, by the rule

$$d\chi^{(m+1)} = d_R\chi^{(m)}.$$

That this is possible follows from the commutation relation $dd_R + d_Rd = 0$: we have, for $m \geqslant 1$,

$$dd_R\chi^{(m)} = -d_Rd\chi^{(m)} = -{d_R}^2\chi^{(m-1)} = 0$$

so the scheme is consistent provided that $dd_R\chi^{(0)} = -d_Rd\chi^{(0)} = 0$.

To make the correspondence with bi-Hamiltonian systems we suppose that $M$ is a Poisson manifold, whose Poisson structure comes from a symplectic form $\omega_0$; that $R$ and $\omega_0$ are such that for every pair of vector fields $X$ and $Y$ on $M$,

$$\omega_0(R(X), Y) = \omega_0(X, R(Y))$$

so that $\omega_1$, defined by $\omega_1(X, Y) = \omega_0(R(X), Y)$, is a 2-form; and that $d\omega_1 = 0$. Then if we set, for $f, g \in C^\infty(M)$,

$$\{f, g\}_1 = \omega_1(X_f, X_g)$$

where $X_f$ is the Hamiltonian vector field corresponding to $f$ with respect to $\omega_0$, then $\{\cdot,\cdot\}_1$ is bilinear over $\mathbb{R}$, skew-symmetric, and satisfies the derivation property

$$\{f, gh\}_1 = g\{f, h\}_1 + \{f, g\}_1 h.$$

Furthermore, it follows from the vanishing of the torsion of $R$, together with the closure of $\omega_1$, that the Jacobi identity holds, so that $\{\cdot,\cdot\}_1$ is a second Poisson bracket on $M$, which is moreover compatible with $\{\cdot,\cdot\}_0$, the Poisson bracket coming from $\omega_0$. Thus in such a case a bi-differential calculus endows $M$ with a Poisson–Nijenhuis structure, that is, with a second Poisson bracket compatible with the first; $R$ is the recursion tensor of the structure.

The construction of the $\chi^{(m)}$, in the case $s = 1$, translates into the terminology of Poisson brackets as follows. We assume that $M$ is such that closed 1-forms are exact. From the definition

$$\{f, g\}_1 = \omega_1(X_f, X_g) = \omega_0(X_f, R(X_g)) = -R(X_g)f = -d_Rf(X_g)$$

the inductive definition of the functions $\chi^{(m)}$ can be expressed as follows:

$$\{\chi^{(m+1)}, \cdot\}_0 = \{\chi^{(m)}, \cdot\}_1.$$

It is easy to show that functions $\chi^{(m)}$ so defined are in involution with respect to both Poisson brackets—we shall outline the proof of a more general result below.

Dimakis and Müller-Hoissen usually impose the initial condition that $d\chi^{(0)} = 0$. However, the scheme will also work with the less restrictive initial condition that $dd_R\chi^{(0)} = -d_Rd\chi^{(0)} = 0$, as they remark and as we remarked above. We will show that, under sufficiently generic conditions, the sum of the eigenfunctions of $R$ satisfies this condition.

Note first that if $X$ is an eigenvectorfield of $R$ with eigenfunction $\lambda$, and if $X'$ is an eigenvectorfield of $R$ with eigenfunction $\lambda'$, then from the symmetry condition on $R$

$$(\lambda - \lambda')\omega_0(X, X') = 0.$$

It follows that $R$ can have at most $n$ functionally independent eigenfunctions, where $\dim M = 2n$. We consider the case in which $R$ has $n$ functionally independent eigenfunctions, the maximum number, such that where the eigenvalues are distinct each is doubly degenerate. It follows from the vanishing of the torsion of $R$ that if the eigenfunctions are $\lambda_a$, $a = 1, 2, \ldots, n$, and $X_a$ is any eigenvector-field corresponding to $\lambda_a$, then

$$X_a(\lambda_b) = 0 \qquad b \neq a.$$

It is clear from dimensional considerations that the two-dimensional eigendistribution corresponding to $\lambda_a$ must contain a one-dimensional subspace $\langle Y_a \rangle$ such that $Y_a(\lambda_a) = 0$; and we may therefore choose a (local) basis of vector fields $\{Y_a, Z_a | a = 1, 2, \ldots, n\}$ such that for each $a$, $\langle Y_a, Z_a \rangle$ is the eigendistribution corresponding to $\lambda_a$, $Y_a(\lambda_a) = 0$, and $Z_a(\lambda_a) = 1$. Now set

$$\chi^{(0)} = \sum_{a=1}^{n} \lambda_a.$$

Then for any eigenvectorfield $X_a$,

$$d_R \chi^{(0)}(X_a) = \sum_{b=1}^{n} d\lambda_b(R(X_a)) = \lambda_a X_a(\lambda_a) = \begin{cases} 0 & \text{if} \quad X_a = Y_a \\ \lambda_a & \text{if} \quad X_a = Z_a. \end{cases}$$

It follows that

$$d_R \chi^{(0)} = \sum_{a=1}^{n} \lambda_a d\lambda_a = \tfrac{1}{2} d\left( \sum_{a=1}^{n} {\lambda_a}^2 \right).$$

The sequence of functions generated in this case can, without essential loss of generality, be taken to be the sums of the powers of the eigenfunctions of $R$, or equivalently the traces of the powers of $R$. We therefore recover the result that the traces of the powers of the recursion tensor of a Poisson–Nijenhuis structure are in involution with respect to both Poisson brackets. (The sequence of functions generated by the scheme of Dimakis and Müller-Hoissen is in principle infinite, but of course only the first $n$ elements of the sequence are functionally independent.)

We can also give a simple example of what Dimakis and Müller-Hoissen call a gauged bi-differential calculus. In a gauged bi-differential calculus the derivations $d$ and $\delta$ are replaced by operators

$$D_d = d + A \qquad D_\delta = \delta + B$$

where in general $A$ and $B$ are square matrices of 1-forms and the operators act on square matrices of forms. The operators have to satisfy the conditions

$${D_d}^2 = {D_\delta}^2 = 0 \qquad D_d D_\delta + D_\delta D_d = 0.$$

In our example the operators act on functions and $D_d = d$; however,

$$D_\delta = d_R + df$$

where $d_R$ is the derivation of type $d$ and degree 1 associated with the type (1, 1) tensor $R$ as before, and $f$ is a function whose properties are to be specified. It is easy to see that $D_d D_\delta + D_\delta D_d = 0$ follows from the fact that $dd_R + d_R d = 0$. If we assume that $R$ has zero torsion, so that ${d_R}^2 = 0$, then the condition that ${D_\delta}^2 = 0$ reduces to $d_R df = 0$. If $f$ satisfies this condition then we have a graded bi-differential calculus.

Following Dimakis and Müller-Hoissen we now have a new scheme for inductively generating a sequence of functions $\chi^{(m)}$, $m = 0, 1, 2, \ldots$:

$$d\chi^{(m+1)} = d_R \chi^{(m)} + \chi^{(m)} df.$$

(The original scheme is of course obtained by setting $f = 0$.) The consistency of this scheme follows from the general theory in [1], but can easily be demonstrated directly; we require that

$$d(d_R \chi^{(m)} + \chi^{(m)} df) = -d_R d\chi^{(m)} + d\chi^{(m)} \wedge df = 0.$$

For $m > 1$ we have

$$-d_R(d_R\chi^{(m-1)} + \chi^{(m-1)}df) + d\chi^{(m)} \wedge df$$
$$= (d\chi^{(m)} - d_R\chi^{(m-1)}) \wedge df - \chi^{(m-1)}d_Rdf = -\chi^{(m-1)}d_Rdf = 0.$$

We shall take for initial function $\chi^{(0)} = 1$: then $\chi^{(1)} = f$, apart from a constant which will be ignored. We now show that the functions so generated are in involution with respect to both Poisson brackets. The rule for generating $\chi^{(m+1)}$, when expressed in terms of Poisson brackets, and with $f$ replaced by $\chi^{(1)}$, is

$$\{\chi^{(m+1)}, \cdot\}_0 = \{\chi^{(m)}, \cdot\}_1 + \chi^{(m)}\{\chi^{(1)}, \cdot\}_0.$$

Assume that $\{\chi^{(i)}, \chi^{(j)}\}_0 = \{\chi^{(i)}, \chi^{(j)}\}_1 = 0$ for all $i, j$ with $1 \leqslant i, j \leqslant m$: we show that the same is true with $m + 1$ in place of $m$. First, for $1 \leqslant i \leqslant m$

$$\{\chi^{(m+1)}, \chi^{(i)}\}_0 = \{\chi^{(m)}, \chi^{(i)}\}_1 + \chi^{(m)}\{\chi^{(1)}, \chi^{(i)}\}_0 = 0.$$

Then

$$0 = \{\chi^{(i+1)}, \chi^{(m+1)}\}_0 = \{\chi^{(i)}, \chi^{(m+1)}\}_1 + \chi^{(i)}\{\chi^{(1)}, \chi^{(m+1)}\}_0$$

whence $\{\chi^{(i)}, \chi^{(m+1)}\}_1 = 0$.

Suppose that we take $f$ to be the sum of the eigenfunctions of $R$, as before. It can then be shown that the functions so generated are the elementary symmetric polynomials in the eigenfunctions of $R$; and these functions are again in involution with respect to both Poisson brackets.

The construction just described appears in a recent paper by Ibort *et al* [2], which is concerned with the so-called Gelfand–Zakharevich bi-Hamiltonian systems and their application to the problem of the separation of variables in the Hamilton–Jacobi equation for Hamiltonians of mechanical type. The proof that the functions $\chi^{(m)}$ are the elementary symmetric polynomials in the eigenfunctions of $R$ may be found there.

A paper containing, among other things, a more detailed discussion of the issues raised above is being prepared by the present authors.

## References

[1] Dimakis A and Müller-Hoissen F 2000 Bi-differential calculi and integrable models *J. Phys. A: Math. Gen.* **33** 957–74

[2] Ibort A, Magri F and Marmo G 2000 Bi-Hamiltonian structures and Stäckel separability *J. Geom. Phys.* **33** 210–23